Fermionic q-oscillators and associated canonical q-transformations

1992 J. Phys. A: Math. Gen. 25 L39

(http://iopscience.iop.org/0305-4470/25/2/002)

J. Phys. A: Math. Gen. **25** (1992) L39–L43. Printed in the UK

LETTER TO THE EDITOR

Fermionic q-oscillators and associated canonical q-transformations

Kazuhiko Odaka

Department of Mathematics and Physics, National Defence Academy, Yokosuka 239, Japan

Received 5 November 1991

Abstract. Within the framework of the R-matrix formalism of the quantum group we deform the algebra of fermionic oscillators and study associated canonical transformations which mix creation and annihilation operators of the deformed oscillators.

A quantum group, that is a q-deformation of Lie groups, can be considered as effecting linear transformations upon quantum spaces whose coordinates are non-commutative and non-anticommutative [1]. The quantum space is the deformation of the configuration space where a particle moves [2], while the coordinates of the quantum space can be also identified with annihilation operators of q-deformed harmonic oscillators (q-oscillator) [3, 4]. Thus, in quantum mechanics on the quantum space the quantum group may appear as q-deformed canonical transformations (canonical q-transformation) under which q-deformed quantum commutation relations are covariant. In fact, there exist already some studies on the canonical q-transformations from the viewpoint of the quantum group [3–5].

The aim of this letter is to extend these studies on canonical q-transformations to the fermionic case containing q-deformed Bogoliubov transformations by using the R-matrix formalism of the quantum group.

We summarize standard quantum mechanics [6] of a system consisting of n fermionic oscillators described by the operators a_k and $a_k^\dagger$ $(k=1,2,\ldots,n)$. The anticommutation relations of the operators are

$$\{a_k, a_l\}=0 \qquad \{a_k^\dagger, a_l^\dagger\}=0 \qquad \{a_k, a_l^\dagger\}=\delta_{kl}. \tag{1a, b, c}$$

These relations (1) are covariant under the infinitesimal transformations

$$\begin{aligned} a_k \to a'_k &= a_k - \mathrm{i}\sum_{m=1}^{n}(\nu_{km}a_m + \lambda_{km}a_m^\dagger) \\ a_k^\dagger \to a_k'^\dagger &= a_k^\dagger - \mathrm{i}\sum_{m=1}^{n}(\nu_{mk}a_m^\dagger + \mu_{mk}a_m) \end{aligned} \tag{2}$$

where the infinitesimal parameters ν_{mk}, λ_{km} and μ_{mk} satisfy

$$\nu_{mk}^* = \nu_{km} \qquad \lambda_{km}^* = \mu_{mk} \qquad \lambda_{km} = -\lambda_{mk}. \tag{3}$$

The infinitesimal transformations (2) and the finite transformations induced by (2) form the group SO($2n$) whose bilinear invariant is given by

$$\Lambda \equiv \sum_{k=1} (a_k^\dagger a_k + a_k a_k^\dagger) = n. \tag{4}$$

For the convenience of the following study, line up the creation and annihilation operators as follows:

$$x^i \equiv a^\dagger_{n+1-i} \qquad x^{n+i} \equiv a_i \qquad (i=1,2,\ldots,n). \tag{5}$$

We introduce the permutation matrix $P^{ij}_{kl} \equiv \delta^j_k \delta^i_l$ and the unit matrix $I^{ij}_{kl} \equiv \delta^i_k \delta^j_l$, so the anticommutation relations become

$$\sum_{k,l} (I^{ij}_{kl} + P^{ij}_{kl}) x^k x^l = \delta^{i\hat{j}} \tag{6}$$

where $\hat{j} \equiv 2n+1-j$. The canonical transformations can be rewritten as

$$x'^i = \sum_{k=1}^{2n} t^i_k x^k \tag{7}$$

where the matrix t^i_k satisfies

$$\sum_{k,l} t^i_k t^j_l \delta^{k\hat{l}} = \delta^{i\hat{j}}. \tag{8}$$

The bilinear invariant reads

$$\Lambda = \sum_{i,j} \delta_{i\hat{j}} x^i x^j. \tag{9}$$

We deform the algebra of a_k and $a^\dagger_k$ by identifying the fermionic coordinates of the quantum space with operators A^i $(i=1,2,\ldots,n)$ of q-oscillators, which correspond to a^i. The anticommutation relations $(1a)$ are deformed as

$$A^{i2} = 0 \qquad A^i A^j + q^{-1} A^j A^i = 0 \qquad (i<j) \tag{10}$$

where q is a complex number. By means of the R-matrix for the quantum group $\mathrm{GL}_q(n)$ the relations (10) can be cast in the form:

$$S^{ij}_{kl} A^k A^l \equiv (I^{ij}_{kl} + q\hat{R}^{ij}_{Lkl}) A^k A^l = 0 \tag{11}$$

where $\hat{R} = PR$ and

$$\begin{aligned} &\hat{R}^{ij}_{Lkl} \equiv \delta^j_k \delta^i_l (1 + (q-1)\delta^{ij}) + (q-q^{-1}) \delta^i_k \delta^j_l \theta_{ji} \\ &\theta_{ji} = \begin{cases} 0 & j>i \\ 1 & j \leq i. \end{cases} \end{aligned} \tag{12}$$

Summation over repeated indices is implicit. Taking the Hermitian conjugate of (11), we have the q-deformed version of $(1b)$

$$A^{\dagger 2}_i = 0 \qquad A^\dagger_j A^\dagger_i + \frac{1}{q^*} A^\dagger_i A^\dagger_j = 0 \qquad (i<j). \tag{13}$$

The q-deformed anticommutation relations between $A^\dagger_j$ and A^i are assumed to be

$$A^\dagger_j A^i = f^i_j + C^{il}_{jk} A^k A^\dagger_l \tag{14}$$

where f^i_j and C^{il}_{jk} are numerical coefficients. Further, we assume that $A^\dagger_j$ and A^j have the properties of creation and annihilation operators, respectively.

The relations (10) should be valid also when multiplied by a Fock state $F(A, A^\dagger)|0\rangle$

$$S^{ij}_{kl} A^k A^l F(A, A^\dagger)|0\rangle = 0 \tag{15}$$

where $|0\rangle$ is the Fock vacuum and $F(A^\dagger, A)$ is a function of A^i and $A_i^\dagger$. Multiplying (15) by $A_m^\dagger$ and using the relations (14),

$$S_{kl}^{ij}A_m^\dagger A^k A^l F(A, A^\dagger)|0\rangle = S_{kl}^{ij}\{f_m^k\delta_n^l + C_{mn}^{kh} f_h^l\}A^n F(A, A^\dagger)|0\rangle$$
$$+ S_{kl}^{ij}C_{mb}^{kh}C_{he}^{ld}A^b A^e(A_d^\dagger F(A, A^\dagger)|0\rangle) = 0. \tag{16}$$

Since $F(A, A^\dagger)|0\rangle$ and $A_d^\dagger F(A, A^\dagger)|0\rangle$ are independent and arbitrary states, we have the consistency conditions

$$S_{kl}^{ij}\{f_m^k\delta_n^l + C_{mn}^{kh} f_h^l\} = 0 \tag{17}$$

$$S_{kl}^{ij}C_{mb}^{kh}C_{he}^{ld}A^b A^e = 0. \tag{18}$$

By use of (10) the condition (18) can be rewritten as

$$\hat{R}_{Lkl}^{ij}C_{mb}^{kh}C_{he}^{ld}A^b A^e = C_{ml}^{ia}C_{ak}^{jd}\hat{R}_{Lbe}^{lk}A^b A^e. \tag{19}$$

This equation is satisfied if the Yang–Baxter relation

$$\hat{R}_{Lkl}^{ij}C_{md}^{kh}C_{he}^{ld} = C_{ml}^{ia}C_{ak}^{jd}\hat{R}_{Lbe}^{lk} \tag{20}$$

holds. A solution of equations (17) and (20) is

$$f_j^i = \delta_j^i f_j$$
$$C_{kl}^{ij} = -\frac{1}{q} f_k \hat{R}_{Lkl}^{ij} f_j^{-1}. \tag{21}$$

The Hermitian conjugate of (14) with (21) should give the same relations, so q and f_i must be real.

We must check the self-consistency of the q-deformed algebra. To this end we prove that for the algebra (10), (13) and (14) with (21) there exist the representations based on the Fock space. The vacuum, the Fock states and the number operators are introduced as

$$A^k|0\rangle = 0 \tag{22}$$

$$(A_1^\dagger)^{n_1}(A_2^\dagger)^{n_2}\dots(A_n^\dagger)^n n|0\rangle \equiv |n_1, n_2, \dots, n_n\rangle \tag{23}$$

$$N_k|n_1, n_2, \dots, n_n\rangle = n_k|n_1, n_2, \dots, n_n\rangle \tag{24}$$

where $n_k = 1$ or 0. Let us multiply $|n_1, n_2, \dots, n_n\rangle$ by A^i or $A_i^\dagger$. Using the q-deformed algebra, we get

$$A_j^\dagger|n_1, n_2, \dots, n_n\rangle = (-q)^{-\Sigma_{j>k}n_k}(1-n_j)|n_1, n_2, \dots, n_j+1, \dots, n_n\rangle \tag{25}$$

$$A^j|n_1, n_2, \dots, n_n\rangle = (-q)^{\Sigma_{j>k}n_k}H(j)|n_1, n_2, \dots, n_j-1, \dots, n_n\rangle \tag{26}$$

where

$$H(j) \equiv n_j f_j \sum_{i=0}^{n-j} G_{j+i}(n_{j+1}, \dots, n_{j+i})$$
$$G_{j+i}(n_{j+1}, \dots, n_{j+i}) \equiv (1-n_{j+i})(q^{-2}-1)\sum_{k=0}^{i-1} G_{j+k}(n_{j+1}, \dots, n_{j+k}) \qquad G_j = 1. \tag{27}$$

Equations (25) and (26) show that $A_j^\dagger$ and A^j are creation and annihilation operators, respectively. The relation between the number operators and $A_j^\dagger A^j$ (no summation) is

$$A_j^\dagger A^j = (2-N_j)\hat{H}(j) \tag{28}$$

where

$$\hat{H}(j) \equiv N_j f_j \sum_{i=0}^{n-j} \hat{G}_{j+i}(N_{j+1}, \ldots, N_{j+i})$$
$$\hat{G}_{j+i}(N_{j+1}, \ldots, N_{j+i}) \equiv (1 - N_{j+i})(q^{-2} - 1) \sum_{k=0}^{i-1} \hat{G}_{j+k}(N_{j+1}, \ldots, N_{j+k}) \qquad \hat{G}_j = 1. \tag{29}$$

The q-deformed relations of A^i and $A_i^\dagger$ are summarized in the following form

$$(I_{kl}^{ij} + q\hat{Q}_{kl}^{ij})X^k X^l = D^{ij} \tag{30}$$

where

$$X^i \equiv A_{n+1-i}^\dagger \qquad X^{n+i} \equiv A^i \qquad (i = 1, 2, \ldots, n) \tag{31}$$

$$\hat{Q}_{kl}^{ij} \equiv F_i F_j \hat{R}_{kl}^{ij} F^k F^l \qquad D^{ij} \equiv F_i F_j C^{ij} \tag{32}$$

$$F^k \equiv (q^{-n+2-k} f_{n+1-k})^{-1/2}$$
$$F^{n+k} \equiv (q^{-2n+k+1} f_k)^{-1/2} \qquad (k = 1, 2, \ldots, n) \tag{33}$$

$$F_i = (F^i)^{-1}. \tag{34}$$

Here,

$$\hat{R}_{kl}^{ij} \equiv q\delta_l^i \delta_k^j \delta^{ij} + q^{-1}\delta_k^j \delta_l^i \delta^{i\hat{j}} + (q - q^{-1})\theta_{ji}\delta_l^j \delta_k^i$$
$$- (q - q^{-1})\theta_{jk} q^{\rho_j - \rho_k} \delta^{i\hat{j}} \delta^{k\hat{l}} + (1 - \delta^{ij})(1 - \delta^{k\hat{l}})\delta_k^j \delta_l^i \tag{35}$$

$$C^{ij} = \delta^{i\hat{j}} q^{\rho_j} \tag{36}$$

with $\hat{j} \equiv 2n + 1 - j$, and $\rho_k \equiv (n-1, n-2, \ldots, 1, 0, 0, -1, -2, \ldots, 1-n)$. $\hat{R}_{kl}^{ij}$ is the $\hat{R}$-matrix for the quantum group $SO_q(2n)$ [7]. Then, it is obvious that the matrix $\hat{Q}_{kl}^{ij}$ satisfies the Yang–Baxter relations.

One can see immediately that the relations (30) are covariant under the action of a linear transformation

$$X'^i = T_k^i X^k \tag{37}$$

where the matrix elements T_j^i are assumed to commutate with X^i, provided they satisfy

$$\hat{Q}_{ik}^{ij} T_a^l T_b^k = T_i^i T_k^j \hat{Q}_{ab}^{lk} \tag{38}$$

$$T_k^i T_l^j D^{kl} = D^{ij}. \tag{39}$$

These are the conditions for matrices to belong to the q-deformed version of the group SO($2n$). These conditions are a little different from ones for the quantum grop $SO_q(2n)$ given in [7], but the difference is not essential. For $f_i \to 1$ and $q \to 1$ the relations (30), the transformations (37) and the condition (39) reduce to (6), (7) and (8), respectively.

The bilinear invariant of the transformations (37), which corresponds to (9), is

$$\Lambda_q \equiv D_{ij} X^i X^j = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{40}$$

where $D_{ij} = F^i F^j C^{ij}$.

In the standard case [6], the generators of the group SO($2n$), which satisfy the Lie algebra, can be given in the bilinear form of the creation and annihilation operators. From the commutation relations between the generators and operator we find the trilinear commutation relations which agree with those proposed by Green in the investigation of the parastatistics.

It is interesting to study how these situations are deformed for the q-deformed oscillators and the quantum group $SO_q(2n)$ given in sections 3 and 4. We are studying this problem and the result will be published elsewhere.

The author is indebted to S Kamefuchi and T Kishi for stimulating discussions. He would like to thank A Hosoya and N Sakai for warm hospitality at the Tokyo Institute of Technology.

References

[1] Manin Y 1987 *Ann. Inst. Fourier* **XXXVII f4** 191
[2] Wess J and Zumino B 1990 *Nucl. Phys.* B(Proc. Suppl.) **18B** 302
[3] Pusz W and Woronowicz S L 1989 *Rep. Math. Phys.* **27** 231
Pusz W 1989 *Rep. Math. Phys.* **27** 349
[4] Chainchian M, Kulish P and Lukierski J 1991 *Phys. Lett.* **262B** 43
[5] Zumino B 1991 *LBL-30120*
[6] Ohnuki Y and Kamefuchi S 1982 *Quantum Field Theory and Parastatistics* (Tokyo: University of Tokyo Press; Berlin: Springer)
[7] Takhtajan T 1989 *Adv. Stud. Pure Math.* **19**